APPLICATIONS OF THE EQUATIONS OF UNSTEADY FLOW IN WATER RESOURCES

by

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INTRODUCTION

The equations of unsteady flow play a very important role in the study of numerous problems involving the movement of water, such as overland flow, flow in rivers and channels, the motion of tides and long waves in oceans, surges in canals and harbors, etc. They consist of the equations for the conservation of mass and of momentum. Although they have been known for a long time, yet mathematical difficulties have prevented the application of the complete equations to the solution of many problems, and approximate methods have been used instead. In recent years, many of these difficulties have been resolved, and the equations have received considerable attention aimed at their application to water resources.

Two areas in which the application of the equations of unsteady flow has been the subject of extensive research in recent years, are (1) determination of runoff hydrographs, and (2) flood routing in rivers and channels. Rigorous methods based on the equations of unsteady flow have been applied to the determination of runoff from drainage areas with simple geometry (Schaake 1965, Liggett and Woolhiser 1967, and others). It is expected that in the near future the runoff hydrographs from watersheds can be determined by procedures based on the mechanics of flow. The second area of concern, flood routing in river channels, is important in the field of water resources because of its role in (1) the operation of flood warning systems so that the time of arrival of the highwater stages could be accurately predicted and the loss of life and property thereby mitigated, and (2) in the design of flood control works, where the discharge and stage at various locations in the channel need to be determined. Reservoir regulation, the determination of the effects of storage and improvement works on river flows are also problems in flood routing. Flood routing will be selected here as an illustration of the application of the equations of unsteady flow. The basic concepts involved in flood routing also hold true for other problems mentioned above. More importantly, however, flood routing, because of its large scope in terms of the times and distances involved, is perhaps the most challenging application of the mathematical equations of unsteady flow to an important physical system.

THEORY

The equations of unsteady flow in an open channel are

$$A(\partial v/\partial x) + v(\partial A/\partial x) + \partial A/\partial t = -q \qquad (1)$$

$$\partial v/\partial t + v(\partial v/\partial x) = g(S_o - S_f) - g(\partial y/\partial x) \qquad (2)$$

In these equations, A is the cross-sectional area of the channel, x is the distance along the channel, v is the velocity in the x- direction, S_o is the channel bottom slope, S_f is the friction slope, y is the depth, q is the lateral inflow to the channel and g is the acceleration due to gravity. Equation (1) expresses the conservation of mass and (2) the conservation of momentum. Equation (2) may also be interpreted as the expression for the conservation of energy. The derivation of equations (1) and (2) is given in standard reference works (Chow 1959, Stoker 1957).

The application of the equations of unsteady flow to a given problem requires that the solution of the equations (1) and (2) be obtained such that the boundary conditions of the problem are satisfied. The boundary conditions for a flood flow problem may be given as the discharge hydrograph at the upper end of a river reach and a rating curve at its downstream end. The solution of the problem then may consist of the changes with time in the values of the stage, discharge and velocity at all locations in the channel. A specific objective of the solution might be to find the magnitude and the time of arrival of the flood peak at all locations in the river.

METHODS OF SOLUTION

From a mathematical point of view, equations (1) and (2) are classified as the first-order nonlinear partial differential equations of the hyperbolic type. Analytical solutions of the equations are not known. Any analytical solution that could be attempted with our present day knowledge of mathematics requires that some of the terms be eliminated from the equations. Although a number of such solutions have been obtained, they are not generally useful. In many problems, the neglected terms play important roles and the solutions obtained at their expense will have little value.

The most promising approach to the solutions of the equations of unsteady flow is the use of numerical methods. Until recently, numerical methods, in order to be feasible for desk calculation, also required that the equations be drastically simplified. The storage routing method, well known in engineering practice, is really a simplified version of equations (1) and (2). In this method, equation (1) is retained but equation (2) is reduced to a storage-discharge relationship. The solutions obtained by the storage-routing method may be quite valid when applied to many reservoirs, but it has been shown that consistently reliable results cannot be obtained by such a simplified version of the equations of unsteady flow. However, accurate and reliable results could be expected from the numerical solution of the complete equations.

The numerical solutions of equations (1) and (2) are obtained in two basic steps. In the first step, the differential equations are represented by finite difference algebraic equations, and in the second step the solutions of the algebraic equations are determined.

In applications to river flows, it is usually desired to determine the values of velocity and stage as functions of time and distance. Thus v and y are taken as the dependent and x and t as the independent variables. The distance x and the time t then become the coordinates of discrete points on the x (x,t) plane. Once y and v are known, the area and discharge can be readily computed, if desired. Various schemes for the representation of the differential equations by difference equations and also for establishing the network of points on the (x,t) plane may be used. In the method of characteristics (Amein 1966), equations (1) and (2) are first transformed into ordinary differential equations, and the points on the (x,t) plane constitute a curvilinear network. It is much more convenient, particularly for natural channels, to obtain solutions at given locations and times. Thus, a fixed rectangular mesh of points is commonly used on the (x,t) plane.

In the following paragraphs, two numerical procedures which are based directly on equations (1) and (2) will be presented. A comparison of the two will shed light on the problems and the promises of the numerical solution of the equations of unsteady flow.

AN EXPLICIT METHOD

A useful result to be expected from the solution of the equations of unsteady flow is the determination of the changing flow situation in a river. Starting out with known conditions at a given time, the equations would provide values of the variables at later times. Fig. 1 shows the network of points on the (x,t) plane. It is assumed that all the variables are known at time t, that is at all points on the row t. It is desired to find the values at later times, $t + \Delta t$, $t + 2\Delta t$, ... and so on.

One method for establishing a numerical procedure for the solution of the equations of unsteady flow is to apply the equations at a point M on the row t and to represent every partial derivative in the equations in the finite difference form. It is important to notice that the point M is chosen on the row t on which the values of the variables are known. The finite difference representation of the partial derivatives in the equations (1) and (2) are given as follows:

$$\left.\begin{aligned} \partial v/\partial t\ (M) &= \frac{v(P) - v(M)}{\Delta t} \\ \partial v/\partial x\ (M) &= \frac{v(N) - v(L)}{2\Delta x} \\ \partial y/\partial t\ (M) &= \frac{y(P) - y(M)}{\Delta t} \\ \partial y/\partial x\ (M) &= \frac{y(N) - y(L)}{2\Delta x} \end{aligned}\right\} \qquad (3)$$

When the finite difference expressions are substituted into equations (1) and (2), two simultaneous algebraic equations are obtained. The algebraic equations are linear and the only unknowns are the values of v and y at the point P. In this manner the solutions are advanced from the row t to a point P on the next row, $t + \Delta t$. Computations may be started at one end of a row and terminated at the other end.

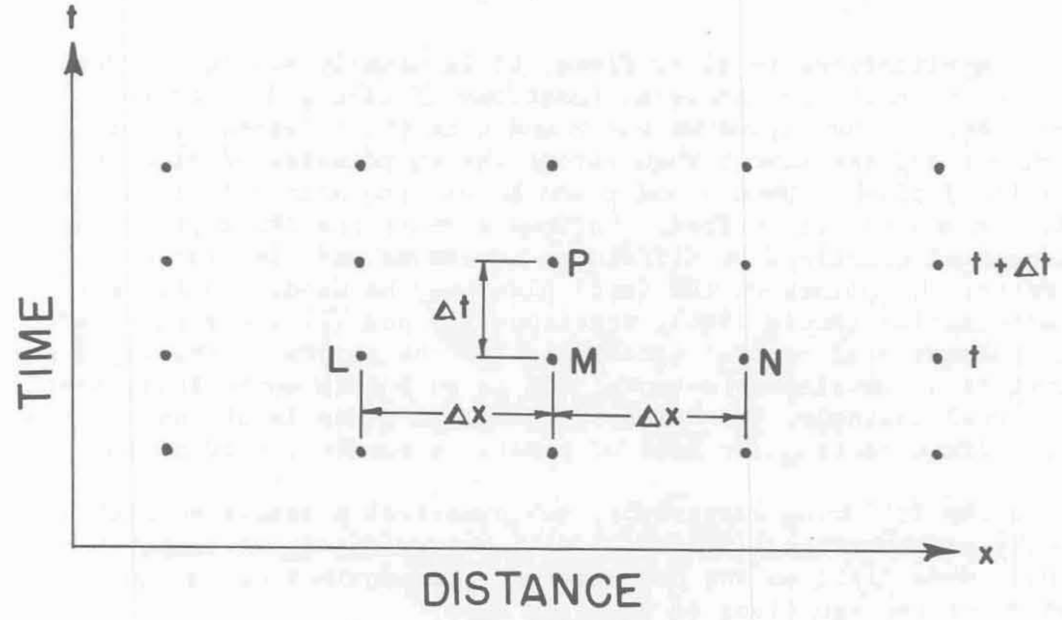

Fig. 1. Network of points for explicit method

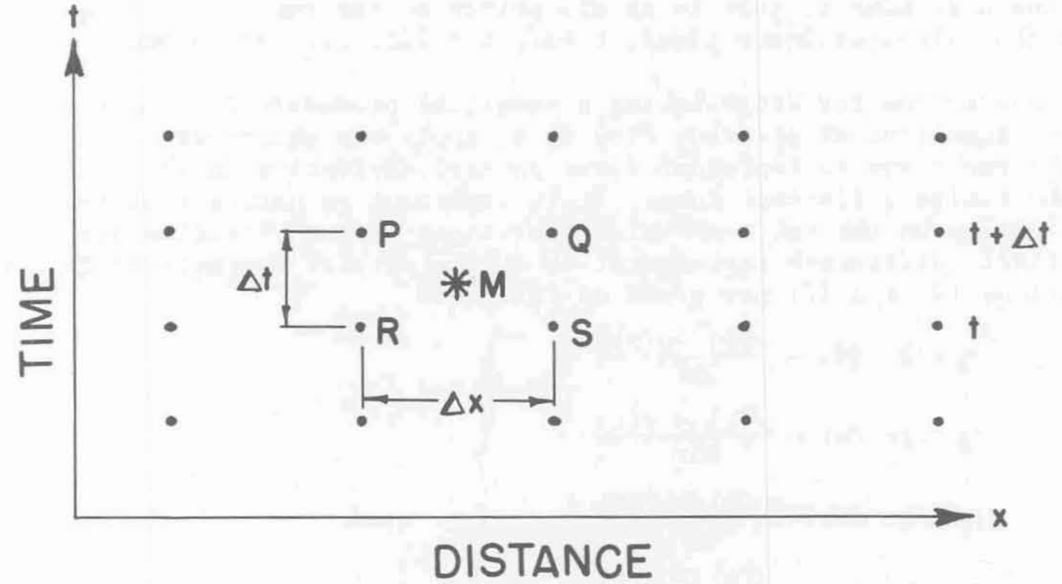

Fig. 2. Network of points for implicit method

This method is based on a very simple representation of the equations of unsteady flow. It is called an explicit method because the finite difference algebraic equations can be solved explicitly for the unknowns. It was first applied by J. J. Stoker and his colleagues (Stoker 1957) to the floods of the Mississippi and Missouri rivers.

The explicit method has one severe disadvantage. The stability of the numerical solution requires that the (Δx, Δt) grid be chosen such that

$$\Delta t \leqslant \frac{\Delta x}{v + \sqrt{gy}}$$

Thus, when Δx = 3 mi., y = 10 ft. and v = 5 ft./sec., Δt must be less than 13 min. When the problem is the routing of a flood of long duration through many miles of a river channel, the computing time required is very long. When this method was applied to the routing of a flood through a 40-mi. reach of the Neuse River in North Carolina, it was found that several hours were needed to complete the problem on the IBM-1410 system. The introduction of the newer breed of computers such as the IBM-360 system has drastically reduced the computing times, but it still would be an expensive method to use for problems involving large distances and long times. In addition, because of the large number of computational steps, the accuracy and reliability of the results suffer from cumulative numerical errors. Perhaps it is for these reasons that the method has not become part of the accepted practice, and instances of failure and disappointment with it have been reported in the literature. However, because of its great simplicity, it would be still useful for laboratory investigations and for applications to small areas.

AN IMPLICIT METHOD

A second procedure for the numerical solution of the equations of unsteady flow would be established by applying the equations to a point M situated midway between the rows (t) and (t + Δt), as shown on Fig. 2. Again, it is assumed that the values of the variables are known on the row (t), and that it is desired to advance the solution to the row (t + Δt). In this scheme, the values of the variables are not known at M, but the variables and the partial derivatives at the point M are expressed in terms of the values at the four corner grid points by the following relations:

$$v(M) = 1/4 \left\{ v(P) + v(Q) + v(R) + v(S) \right\} \tag{4}$$

$$y(M) = 1/4 \left\{ y(P) + y(Q) + y(R) + y(S) \right\} \tag{5}$$

$$S_f(M) = 1/4 \left\{ S_f(P) + S_f(Q) + S_f(R) + S_f(S) \right\} \tag{6}$$

$$\partial v/\partial x\ (M) = \frac{v(Q) + V(S) - v(P) - v(R)}{2.\Delta x} \tag{7}$$

$$\partial y/\partial x(M) = \frac{y(Q) + y(S) - y(P) - y(R)}{2.\Delta x} \tag{8}$$

$$\partial v/\partial t(M) = \frac{v(P) + v(Q) - v(R) - v(S)}{2.\Delta t} \tag{9}$$

$$\partial y/\partial t(M) = \frac{y(P) + y(Q) - y(R) - y(S)}{2.\Delta t} \tag{10}$$

When the expressions on the right-hand sides of (4)through(10) are substituted into equations (1) and (2), two algebraic equations are obtained for the point M. The two equations are, however, nonlinear. Furthermore, the two equations contain four unknowns, the unknowns being the values of the depth and velocity at the points P and Q. But if the two simultaneous equations are written for all points on the row $(t + \Delta t)$, the number of equations obtained will be equal to the number of unknowns. The use of this procedure requires that a large number of simultaneous algebraic equations be solved. This method of approach is called implicit, because the unknowns occur implicitly in the algebraic equations. Its main advantage is that it does not suffer from the restrictive stability condition of the explicit method so that large values for Δt, say one or more hours, may be used in the computations.

The difficulties of dealing with nonlinear systems of equations have discouraged effective use of the implicit method. Recently (Amein 1968) it was found that Newton's iterative method can be successfully used to the solution of the system. The iteration scheme was found to be rapidly convergent for flood routing, and problems requiring long computing times by other methods could be quickly solved by this method. A flood flow with a duration of several days may be routed through a long channel in a few minutes. The method has been successfully applied to floods in natural channels, to tidal flows in inlets and other problems.

ILLUSTRATIONS

Two figures are presented for purposes of illustration. Both figures display the propagation of a flood in a long channel and are the solutions of a problem suggested by Thomas (1934). In Fig. 3 the solution of the problem by the storage-routing method is presented. It is seen that the time of arrival and the magnitude of the flood peak depend on the choice for values of Δx and Δt. The unreliability of the simplified methods for the investigation of complex phenomena is clearly demonstrated. In Fig. 4, the solution of the same problem by the complete equations, using the implicit method, is shown. It is seen that identical results may be obtained by various combinations of Δx and Δt. The solution of the problem by the explicit method is in close agreement with that obtained by the implicit method. However, the values of Δt used in the explicit method were 0.05, 0.10 and 0.15 hr. in contrast to the values of 0.5, 1.0, 2.0 and 3.0 hr. used in the implicit method. Furthermore, the solution of the problem could not be completed by the explicit method for the higher Δt's because the numerical procedure became unstable.

This example provides us with a perspective on the progress made in the use of the equations of unsteady flow. It is seen that until a few years ago, it was not feasible to attempt the solution of problems by the complete equations, and approximate methods of somewhat dubious validity

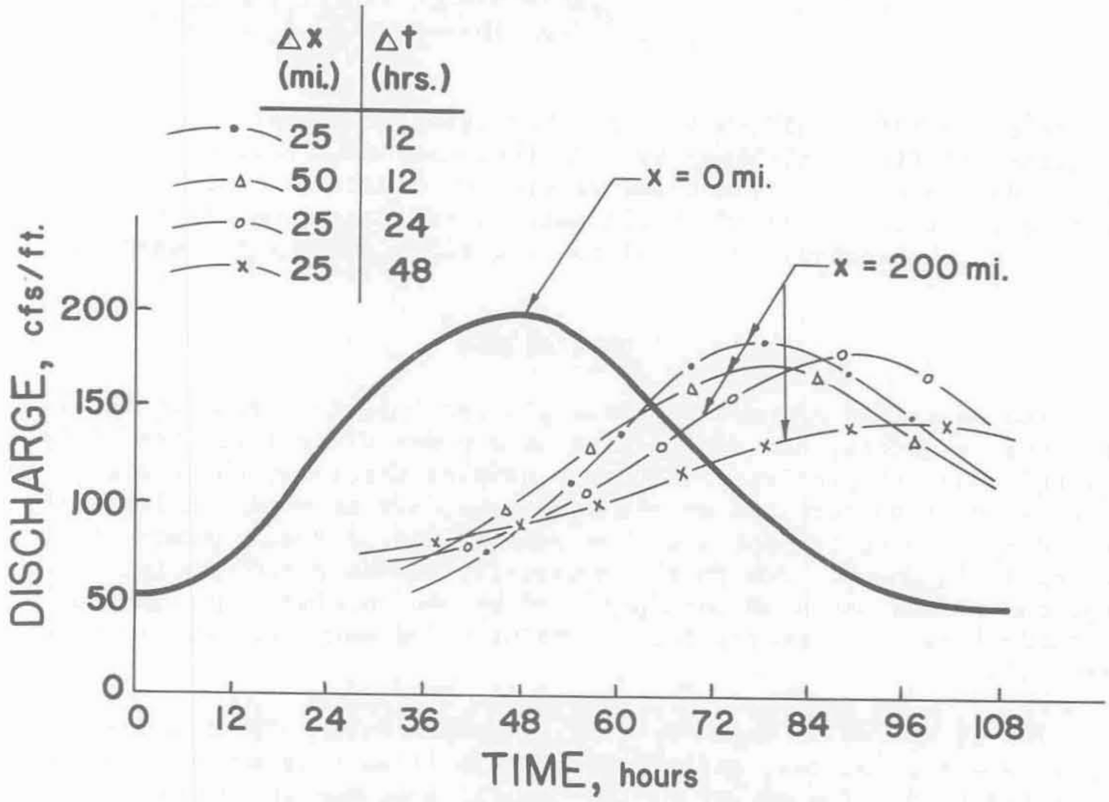

Fig. 3. Flood routing by storage-routing method

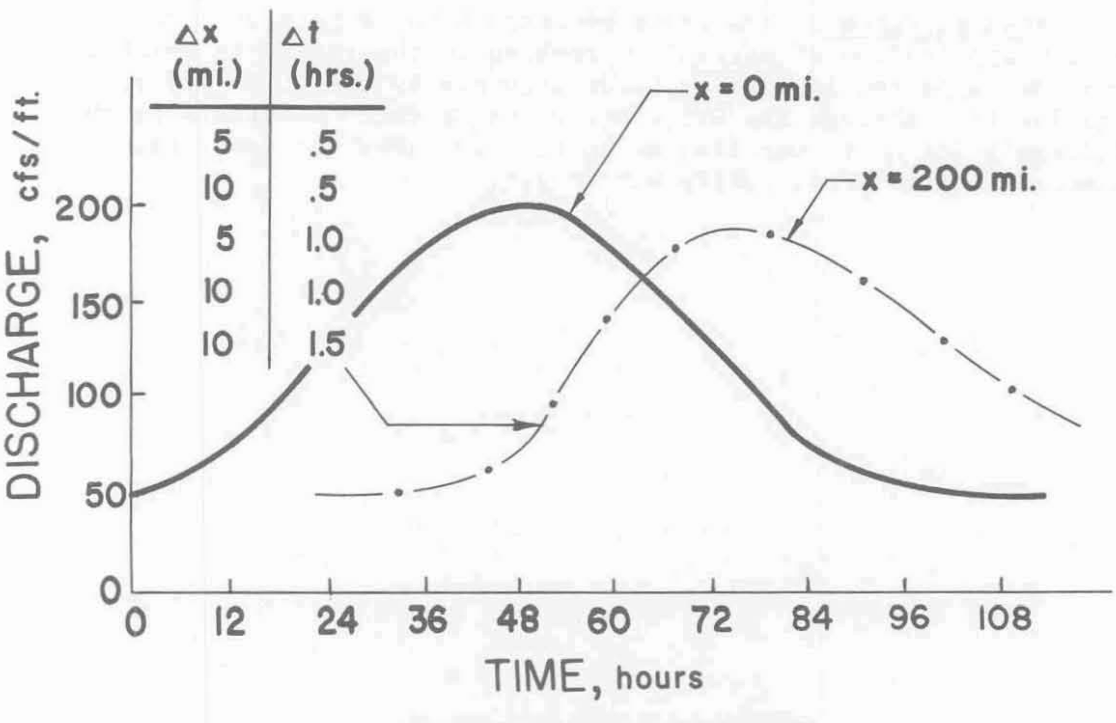

Fig. 4. Flood routing by implicit method

had to be employed. The advent of the digital computer made it possible to attack difficult problems by numerical methods. However, much research was needed, and still is needed, before the equations could be used for practical purposes. As efficient methods and faster machines become available, the analysis of problems of greater and greater complexity can be undertaken.

CONCLUSIONS

The equations of unsteady flow play an important role in the field of water resources, and can be used as a powerful tool for the study of a large class of problems. Although simpler theories, which are in many cases simplified versions of the equations, are commonly employed for providing answere to problems, the reliability of such answers can be seriously in doubt. Due to the complexity of the phenomena involved, a rigorous theory, such as that provided by the complete equations of unsteady flow, is required for the solution of many problems in water resources.

The equations of unsteady flow have been used only in a limited way in water resources, and their potentialities have not been as yet fully realized. The advent of the digital computer and the use of numerical methods have opened opportunities for their utilization. As fast and efficient methods are devised and placed within reach of workers in water resources, great progress in the solution of complex problems is to be expected.

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