# Quick Estimation of Parameters of Muskingum Method of Flood Routing

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## INTRODUCTION

Since its development around 1934 (Gilcrest, 1950; Chow, 1959) by McCarthy (1938) the Muskingum method has been one of the most popular methods for routing flood waves in rivers and channels. It is, therefore, not surprising that many investigators have studied the method and the implications involved in its use (Gilcrest, 1950; Nash and Farell, 1955; Linsley, Kohler and Paulhus, 1958; Chow, 1959; Nash, 1959a; Carter and Godfrey, 1960; Kulandaiswamy, 1966; Overton, 1966; Diskin, 1967; Venetis, 1969; Cunge, 1969; Gill, 1977, 1978, 1979a, 1979b; Koussis, 1978; Ponce and Yevjevich, 1978a, 1978b; Ponce, 1979; Meehan, 1979; Stephensen, 1979). Most agree that the effectiveness of this method depends on the accuracy with which parameters are estimated. Most frequently the parameters are determined graphically (Linsley, Kohler and Paulhus, 1975; Viessman, Knapp, Lewis and Harbaugh, 1977). Although the graphical method is generally satisfactory, it certainly is not the most convenient one. Furthermore, it entails a considerable element of subjectivity, and is slow to work with. Therefore, a more objective method of parameter estimation might be preferable. This paper determines the Muskingum parameters using (1) least squares method, (2) method of moments, (3) method of cumulants, (4) graphical method and (5) direct optimization. An example is worked out to evaluate the efficiency of each method. It is shown that it is far more convenient to estimate the parameters by either of the four other methods, and that there may exist more than one set of parameters leading to comparable results.

## MUSKINGUM METHOD OF FLOOD ROUTING

The Muskingum method consists of a spatially lumped form of a continuity equation and a linear storage-discharge relationship for a specified river reach which can be written respectively as:

$$I = 0 + \frac{ds}{dt} \tag{1}$$

$$S = k[xI + (1-x)0] \tag{2}$$

where I is the rate of inflow, 0 the rate of outflow, S the storage, t the time, and k and x are the routing parameters. Physically speaking, k is the average reach travel time and x the coefficient used to weigh the relative effects of inflow and outflow on reach storage. x can take values from 0 to .5.

The Muskingum method is a numerical solution of Eqs. (1) - (2) at a time $t_n$ and may be expressed as follows:

$$0_n = C_0 I_n + C_1 I_{n-1} + C_2 0_{n-1} \tag{3a}$$

where

$$C_0 = (-kx + 0.5\ \Delta t)/C_3 \tag{3b}$$

$$C_1 = (kx + 0.5\ \Delta t)/C_3 \tag{3c}$$

$$C_2 = (k - kx - 0.5\ \Delta t)/C_3 \tag{3d}$$

$$C_3 = k - kx + 0.5\ \Delta t \tag{3e}$$

$$t = t_n - t_{n-1} \tag{3f}$$

Further,

$$C_0 + C_1 + C_2 = 1 \tag{4}$$

and

$$\Delta t \geqslant 2kx \tag{5}$$

It is thus seen that an accurate determiniation of the parameters x and k is central to the usefulness of the Muskingum method.

## PARAMETER ESTIMATION

The storage S in Eq. (2) is the absolute storage. In practice, it is the relative storage that is normally available. Thus Eq. (2) can be modified to incorporate the difference C between relative and absolute storages:

$$S = k[xI + (1-x)0] + C \tag{6}$$

Thus the problem is reduced to estimating, k, x and C.

### Least Squares Method

This method is based on minimizing the sum of squares of deviations between observed storage and computed storage for a given inflow-outflow sequence. Mathematically,

$$E = \sum_{j=1}^{N} (S_0(j) - S_c(j))^2 => \min \tag{7}$$

where $S_0(j)$ is the observed storage for the j-th time interval, $S_c(j)$ the estimated storage for the j-th time interval and N the number of data or times of observations. E is the error function to be minimized. Equation (7) can be written as (dropping j for brevity),

$$E = \sum_{j=1}^{N} (S_0 - kxI - k(1-x)\,0 - C)^2 => \min \tag{8}$$

Thus we obtain:

$$B_j = \frac{y_1 z_2 - z_1 y_2}{z_2 y_3 - y_2 z_3} \tag{9a}$$

$$A = \frac{y_1}{y_2} - B\frac{y_3}{y_2} \tag{9b}$$

$$C = (\Sigma S_o - A\Sigma I - B\Sigma O)/N \tag{9c}$$

where

$$y_1 = \Sigma S_o I - (\Sigma S_o \Sigma I)/N$$
$$y_2 = \Sigma I^2 - (\Sigma I)^2/N$$
$$y_3 = \Sigma OI - \Sigma O\Sigma I/N$$
$$z_1 = \Sigma S_o O - (\Sigma S_o \Sigma O)/N$$
$$z_2 = \Sigma IO - (\Sigma I\Sigma O)/N$$
$$z_3 = \Sigma O^2 - (\Sigma O\Sigma O)/N$$
$$A = kx$$
$$B = k(1-x)$$

Therefore,

$$k = A + B \tag{10a}$$
$$x = A/(A + B) \tag{10b}$$

Thus k, x and C can be determined objectively and conveniently. The calculation involved in the method can be easily carried out on a small desk calculator.

**Method of Moments**

This method is based on determining first the instantaneous unit hydrograh (IUH) of the Muskingum reach and then determining the moments of IUH. If we combine Eq. (1) with Eq. (2) or Eq. (6), a linear ordinary differential equation is obtained:

$$k(1-x)\frac{dO}{dt} + O = I + kx\frac{dI}{dt} \tag{11}$$

It is easy to show that the IUH h(t) of Eq. (11) due to a unit impulse $\delta(t)$ of inflow (Nash, 1959a; Kulandaiswamy, 1966; Diskin, 1967; Venetis, 1969; Dooge, 1973) is

$$h(t) = \frac{e^{-t/(k(1-x))}}{k(1-x)} - \frac{x}{(1-x)}\delta(t) \tag{12}$$

Furthermore, if we take moments of Eq. (12) then

$$U_1^0(h) = k \tag{13}$$
$$U_2(h) = (1-2x)k \tag{14}$$

where $U_1^0(h)$ is the first moment of h(t) about the origin and $U_2(h)$ the second moment of h(t) about its centroid. These moments of IUH can be determined from the moments of I and O using the Nash Theorem of moment (Nash, 1959b). Thus we obtain:

$$U_1^0(O) = U_1^0(I) + U_1^0(h) \tag{15}$$
$$U_2(O) = U_2(I) + U_2(h) \tag{16}$$

where U(.) denotes the moment of (.), its subscript denotes the order of U, its superscript denotes the moment about the origin and no superscript denotes the moment about the centroid, $U_1^0$. Thus k and x can be obtained objectively and conveniently by computing the first two moments of inflow and outflow - one about the origin and the other about the centroid.

**Method of Cumulants**

It is easy to show that

$$K_1 = U_1^0(h) = k \tag{17}$$
$$K_2 = U_2(h) = (1-2x)k \tag{18}$$

where $K_1$ and $K_2$ are the first and second cumulants of IUH in Eq. (12). Using the theorem of cumulants equations analogous to Eqs. (15) - (16) can be expressed for $K_1$ and $K_2$ as

$$K_1(O) = K_1(I) + K_1(h) \tag{19}$$
$$K_2(O) = K_2 + K_2(I) + K_2(h) \tag{20}$$

where $K_i(.)$ represents i-th cumulant of the quantity (.). Thus an objective determination of k and x is possible using the method of cumulants. It should be noted that in this particular case the methods of cumulants and moments are equivalent.

**Graphical Method**

This method consists in choosing x such that the loop resulting from the plot of S versus xI + (1-x)O becomes as close to a straight line as possible. The slope of the straight line fitted through the loop gives k. Obviously, this method, as practiced, results in a trial and error procedure.

However, it we closely follow the steps involved in the graphical method and the result obtained therefrom, it becomes immediately clear that the least squares method is a numerical expression of the graphical method; hence, the two methods are equivalent. Therefore, the least squares method should constitute a natural replacement for the graphical method.

In graphical method we try to choose x such that the correlation coefficient r between S and xI + (1-x)O is maximum. It is well known that

$$r = \frac{COV(S, Q)}{\sigma_S \sigma_Q} \tag{21}$$

where COV(S,Q) is covariance of S and Q, $\sigma_S$ standard deviation of $S_o$, $\sigma_Q$ standard deviation of Q, and Q = xI + (1-x)O. For computation,

$$r = \frac{\Sigma SQ}{(\Sigma S^2 \Sigma Q^2)^{0.5}} \tag{22}$$

$$k = \frac{r\sigma_S}{\sigma_Q} \tag{23}$$

$$C = \overline{S} - k\overline{Q} \tag{24}$$

where $\overline{S}$ is mean of S and $\overline{Q}$ mean of Q. Thus the problem of determining k, x and C is one of maximizing r as a function of x. This is the same that is done in the least squares method.

**Direct Optimization**

This method determines directly the routing coefficients $C_0$, $C_1$, and $C_2$ without estimating k and x, based on minimizing the difference between observed hydrograph and computed hydrograph. The difference can be expressed by an error function defined in a least squares sense or differently. In this study we will employ the least squares error function.

There are, in fact, only two unknowns since the third is known from Eq. (4). If we choose $C_1$ and $C_2$ to be unknowns then

$$x = (C_1 + 0.5\,C_2 - 0.5)/(C_1 + C_2) \tag{25}$$
$$k = \Delta t\,(C_1 + C_2)/(1 - C_2) \tag{26}$$

Further,

$$C_1(I_n - I_{n-1}) + C_2(I_n - O_{n-1}) = I_n - O_n \tag{27}$$

Equation (27) can be used to construct the least squares error function in a manner similar to that of Eq. (8). If we define

$$R_n = I_n - O_n$$
$$F_n = I_n - I_{n-1}$$
$$G_n = I_n - O_{n-1}$$

then

$$R_n = C_1F_n + C_2G_n$$

The error function (dropping the subscript n for brevity) then follows:

$$E = \sum_{n=1}^{N} (R_o - R_c)^2 \Rightarrow \min \tag{29}$$

where the subscripts 0 and e denote observed and estimated R respectively. Substituting Eq. (28) into Eq. (29), then differentiating Eq. (29) once with respect to $C_1$ and once with respect to $C_2$ and equating each time to zero, we obtain:

$$\Sigma R_0 F = C_1 \Sigma F^2 + C_2 \Sigma FG \tag{30}$$
$$\Sigma R_0 G = C_1 \Sigma FG + C_2 \Sigma G^2 \tag{31}$$

Solving Eqs. (30) - (31) for $C_1$ and $C_2$,

$$C_1 = (\Sigma R_0 F \Sigma G^2 - \Sigma R_0 G \Sigma FG)/DET \tag{32}$$
$$C_2 = (\Sigma R_0 F \Sigma F^2 - \Sigma R_0 F \Sigma FG)/DET \tag{33}$$

where

$$DET = \Sigma G^2 2 \Sigma F^2 - (\Sigma FG)^2$$

Thus, once $C_1$ and $C_2$ are determined, k and x can be obtained using Eqs. (25) - (26). The amount of calculation involved in determining $C_0$, $C_1$ and $C_2$ is so small that it can be easily performed on a small desk calculator.

## APPLICATION

An example was worked out to estimate k and x by the above five methods. The example data is from Linsley, Kohler and Paulhus (1958). The parameters k and x were determined for this data as shown in Table 1. It is clear that both least squares and graphical methods yield the same parameter values. The methods of moments and cumulants both yield the same parameter values, but lead to a significantly different value of x when compared with the first two methods. The direct optimization yields even more different values of x and k.

Figure 1 plots observed storage versus xI + (1-x)0 for each of the four methods. It is evident that the three loops in the figure are not as far apart as their corresponding values of k and x might suggest. Indeed the loops corresponding to the graphical and least squares methods with and without C are quite close to each other.

A comparison of errors, defined as (observed flow-computed flow)/observed flow, is given in Table 2. Observed and computed outflows are compared for each method in Fig. 2. Although the five methods yield satisfactory results, the graphical and least squares methods are more accurate than the methods of cumulants and moments. The direct optimization appears to be the most accurate of all. Two points appear to emerge from analysis of the above data. First, the five methods are comparable in their results on the whole. Second, although k and x are significantly different for these methods, they lead to more or less comparable results. This suggests that there might exist more than one set of the parameters k and x for the Muskingum flood routing method. Comparability of the results indicates that there is no particular reason to use the trial and error graphical method.

**Table 1. k and x values for the example data of Linsley, Kohler and Paulhus (1958).**

| Method | x | k (days) | $C_3$ ($M^3$) |
|---|---|---|---|
| When C=0 | | | |
| Graphical | 0.25 | 0.636 | |
| Least Squares | 0.25 | 0.636 | |
| Moments | 0.446 | 0.61 | |
| Cumulants | 0.446 | 0.61 | |
| Optimization | 0.16 | 0.731 | |
| When C≠0 | | | |
| Graphical | 0.272 | 0.78 | -162391.8 |
| Least Squares | 0.272 | 0.78 | -162391.8 |

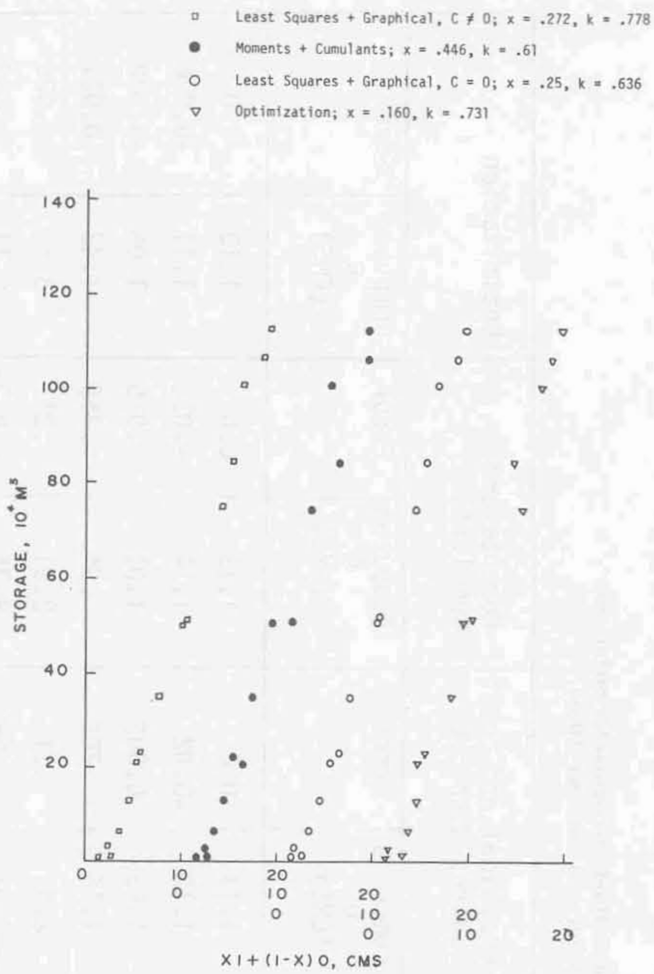


**Figure 1. Storage versus weighted flow for different methods of parameter estimation for data of Linsley, Kohler and Paulhus (1975).**

## CONCLUSIONS

The least squares method is a numerical expression of the graphical method. Therefore, there exists little reason to use the graphical method as practiced currently. The amount of computation involved in the method of cumulants and moments, least squares method and the direct optimization is small enough to be easily managed by a small desk calculator.

## ACKNOWLEDGEMENTS

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Table 2. Comparison of methods of parameter estimation for data of Linsley, Kohler and Paulhus (1958).

| t (days) | I (CMS) | Observed O (CMS) | Graphical + Least Squares with C=0 | | Graphical and Least Squares with C≠0 | | Moments and Cumulants | | Optimization | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | Computed O (CMS) | Error | Computed O (CMS) | Error | Computed O (CMS) | Error | Computed O (CMS) | Error |
| 0 | 1.13 | 1.13 | 1.13 | 0.0 | 1.13 | 0.0 | 1.13 | 0.0 | 1.13 | 0.0 |
| 0.5 | 0.99 | 1.10 | 1.14 | -.02 | 1.13 | -0.02 | 1.14 | -.031 | 1.11 | -0.006 |
| 1 | 1.05 | 1.05 | 1.04 | .010 | 1.05 | 0.002 | 1.01 | .035 | 1.05 | -0.002 |
| 1.5 | 3.54 | 1.47 | 1.36 | .079 | 1.16 | .209 | .94 | .359 | 1.43 | 0.027 |
| 2 | 9.63 | 3.68 | 3.61 | .017 | 2.91 | .21 | 2.91 | .21 | 3.59 | .025 |
| 2.5 | 16.28 | 8.13 | 8.58 | -.056 | 7.33 | .098 | 8.35 | -.027 | 8.11 | .002 |
| 3 | 20.44 | 13.37 | 14.40 | -.077 | 13.01 | .027 | 14.92 | -.116 | 13.48 | -.008 |
| 3.5 | 20.95 | 17.67 | 18.62 | -.054 | 17.58 | .005 | 19.59 | -.109 | 17.58 | .005 |
| 4 | 19.06 | 19.14 | 19.99 | -.044 | 19.56 | -.022 | 20.82 | -.088 | 19.24 | -.005 |
| 4.5 | 12.91 | 18.07 | 18.59 | -.028 | 18.97 | -.050 | 19.57 | -.083 | 18.19 | -.007 |
| 5 | 9.06 | 16.25 | 14.20 | .126 | 15.08 | .072 | 14.07 | .134 | 14.54 | .105 |
| 5.5 | 6.93 | 11.16 | 10.40 | .068 | 11.30 | -.012 | 9.90 | .113 | 11.04 | .010 |
| 6 | 5.44 | 8.69 | 7.83 | .099 | 8.56 | .016 | 7.44 | .144 | 8.44 | .029 |
| 6.5 | 4.08 | 6.65 | 6.02 | .096 | 6.58 | .011 | 5.79 | .129 | 6.49 | .024 |
| 7 | 3.34 | 5.1 | 4.59 | .099 | 5.02 | .016 | 4.37 | .143 | 4.98 | .022 |
| 7.5 | 2.69 | 4.02 | 3.65 | .092 | 3.96 | .015 | 3.52 | .124 | 3.93 | .022 |
| 8 | 2.26 | 3.23 | 2.93 | .090 | 3.16 | .020 | 2.83 | .123 | 3.15 | .025 |
| 8.5 | 1.9 | 2.63 | 2.43 | .078 | 2.60 | .014 | 2.37 | .102 | 2.58 | .02 |
| 9 | 1.58 | 2.18 | 2.02 | .072 | 2.15 | .012 | 1.98 | .092 | 2.14 | .02 |
| 9.5 | 1.42 | 1.81 | 1.70 | .061 | 1.80 | .008 | 1.65 | .088 | 1.79 | .011 |
| 10 | 1.19 | 1.56 | 1.48 | .052 | 1.55 | .003 | 1.46 | .062 | 1.54 | .011 |

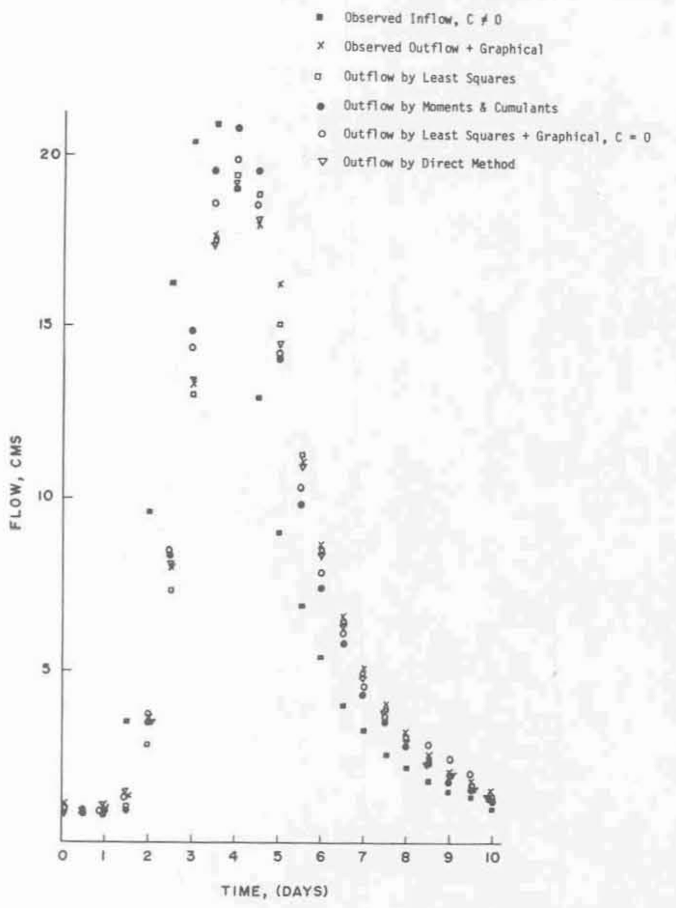


**Figure 2. Flood routing by Muskingum method for different methods of parameter estimation for data of Linsley, Kohler and Paulhus (1975).**


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